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Back Reaction of Charged Black Hole

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The existence of the Hawking radiation of the black hole surely affects space-time. Here, using a thermodynamic approach, to avoid the difficulty of finding the energy-momentum tensor, we obtain expressions for the energy and entropy of the *Reissner–Nördström* black hole (RNBH), and provide a better way of solving the backreaction of more complex black holes.

1. INTRODUCTION

Bekenstein (1973) was the first to put forward a thermodynamic approach to black holes, but it was not until one year later that Hawking found that a black hole has radiation when the theory of quantum fields is used in curved space-time. Only at this time did people begin to consider the black hole as a thermodynamic system to which the four thermodynamic laws applied (Hawking, 1974; Bardeen, *et al.*, 1973). The existence of the Hawking radiation of a black hole surely affects space-time (so-called backreaction). For further developement of black hole thermodynamics, we must consider this backreaction. York (1985) first considered this backreaction, but his approach—a dynamical approach—is useful only in the simplest situation, a Schwarzschild black hole, and does not work for even slightly more complicated situations. However, by studying the backreaction of a black hole with a thermodynamic approach we avoid the difficulty of finding the energy-momentum tensor. After considering the backreaction, the expressions for the black hole energy and entropy are easily given.

The arrangement of this paper is as follows. In Section 2, we calculate the expressions for the energy and entropy of the Reissner–Nördström black hole (RNBH), considering the effect of the Hawking radiation. In Section 3,

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we obtain the equilibrium temperature of the thermodynamic system. Section 4 contains a summary. We use the system of units $c = \hbar = K_B = 1$.

2. THE EFFECT OF THE HAWKING RADIATION ON THE ENERGY AND ENTROPY OF A BLACK HOLE

Let us put the black hole and its Hawking radiation field into an insulated adiabatic cavity, to which a sufficiently long tube with a piston on the other end is connected, so that at the end of the tube the space is asymptotically flat (Fig. 1). When the black hole is taken to be a thermodynamic system made up of a total system with its outside, it obeys thermodynamic laws (Unruh and Wald, 1982). Such a thermodynamic system can be treated as composed of a naked black hole, a two-dimensional thermodynamic membrane (which coincides with the horizon of the black hole), and the radiation in flat space-time.

Supposing the RNBH does not produce the radiation, the total energy E, total entropy S, and the total free energy F of the thermodynamic system are as follow:

$$E = E_0 + M_r \tag{1}$$

$$S = S_0 + S_r \tag{2}$$

$$F = F_0 + F_r \tag{3}$$

where M_0, S_0, and F_0 are the energy, the entropy, and the free energy of the RNBH respectively, and M_r, S_r, and F_r are the energy, the entropy, and the free energy of the radiation field, respectively.

However, the existence of the Hawking radiation of the black hole surely affects space-time. Let us suppose the effect is to change the energy, the entropy, and the free energy by E_H, S_H, and F_H. After considering the effect of the radiation, we obtain the total energy, the total entropy, and the total free energy of the thermodynamic systyem:

$$E_T = M_0 + E_H + M_r \tag{4}$$

$$S_T = S_0 + S_H + S_r \tag{5}$$

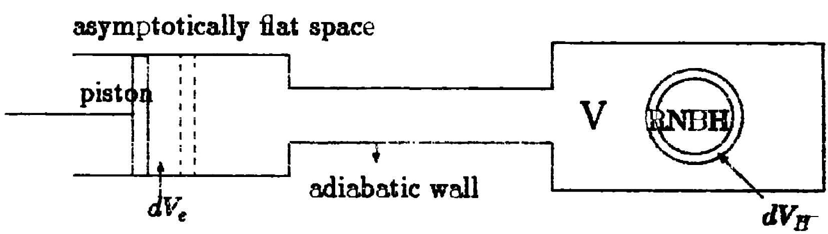

Fig. 1.

$$F_T = F_0 + F_H + F_r \tag{6}$$

When this thermodynamic system is in thermal equilibrium, the free energy F, the entropy S, and the energy of the black hole should give

$$dF = -S\,dT - Q\,dV_0 + \sigma\,dA \tag{7}$$

where $S = S_0 + S_H$ and $F = F_0 + F_H$; V_0 is the potential of the naked black hole at the event horizon, A is the area of the event horizon, Q is the charge number of the black hole. T is the equilibrium temperature of the system, where

$$F_H = \sigma A \tag{8}$$

From (7), we have

$$dF_0 = -S_0\,dT + Q\,dV_0 \tag{9}$$

$$dF_H = -S_H\,dT + \sigma\,dA \tag{10}$$

From (10) and (8), we have

$$S_H = -\left(\frac{\partial F_H}{\partial T}\right)_A = -A\left(\frac{\partial \sigma}{\partial T}\right)_A \tag{11}$$

Supposing δV_H is the change of the radiation field volume caused by the expansion and contraction of the black hole horizon, and δV_e is the change of the radiation field volume caused by the piston motion because of the existence of Hawking radiation. Then the total change of the radiation field volume is

$$\delta V = \delta V_e + \delta V_H \tag{12}$$

According to the first law of thermodynamics, the system in the cavity should obey

$$\delta(M + M_r) = -P\,\delta V_e \tag{13}$$

$$\delta M = T\,\delta S + V_0\,\delta Q + \sigma\,\delta A \tag{14}$$

$$\delta M_r = T\,\delta S_r - P\,\delta V - V_0\,\delta Q \tag{15}$$

during an adiabatic process.

Here M and M_r are the energy of the black hole and radiation field, respectively, with backreaction considered, P is the pressure of the radiation field flat space-time, and $V_0(-\delta Q)$ is the variation of the internal energy of the radiation field caused by the variation of its net charge.

In an asymptotically flat area

$$P = \frac{1}{3} aT^4$$

The total entropy of the system is

$$S_T = S + S_r \tag{16}$$

Thus

$$\delta S_T = \delta S + \delta S_r \tag{17}$$

During the adiabatic process the system should obey

$$T\delta S_T = 0 \tag{18}$$

From (13)–(15) and (18), we get

$$P\,\delta V_e - P\,\delta V + \sigma\,\delta A = 0 \tag{19}$$

or

$$-P\,\delta V_H + \sigma\,\delta A = 0$$

$$\sigma = P\frac{\delta V_H}{\delta A} = \frac{1}{3} aT^4 \frac{\delta V_H}{\delta A} \tag{20}$$

We notice that

$$\sigma = \sigma(T, v), \qquad v = \frac{Q}{r_h} \tag{21}$$

where r_h is the event horizon location of the RNBH, so we can write

$$\frac{\delta V_H}{\delta A} = -f(T, v) \tag{22}$$

We analyze the form of the function $f(T, v)$ as follows:

1. When $Q \to 0$, $f(T, v)$ becomes the result for the Schwarzschild black hole (Huang *et al.*, 1993).

2. Dimensional analysis yields

$$f(T, v) = \frac{\lambda}{T} + \sum_{n=1}^{\infty} \frac{\beta_n v^n}{T^{n+1}} \tag{23}$$

where λ, β are dimensionless real constants.

3. When the signs of v and Q change, the sign of $f(T, v)$ or $\delta V_H/\delta A$ does not. So $n = 2m$, $m = 1, 2, 3, \ldots$ When $T \to 0$, we should have $\sigma \to 0$. So the only choice is $m = 1$. From the above analysis, we obtain the form of the function $f(T, v)$ as follows:

$$f(T, v) = \frac{\lambda}{T} + \frac{\beta v^2}{T^3} \tag{24}$$

From (20), we get

$$\sigma = -\frac{1}{3} aT^3 \left(\lambda + \beta \frac{v^2}{T^2}\right) \tag{25}$$

$$S_H = -A\left(\frac{\partial \sigma}{\partial T}\right)_A = \left(aT^2\lambda + \frac{1}{3} av^2\beta\right)A \tag{26}$$

Thus, if we consider the backreaction, the energy (mass) and the entropy of the RNBH are, respectively,

$$M = M_0 + \sigma A + TS_H = M_0 + \frac{2}{3} \lambda a T^3 A \tag{27}$$

$$S = S_0 + \left(\lambda a T^2 + \frac{1}{3} \beta a v^2\right)A \tag{28}$$

where M_0 and S_0 are the energy (mass) and the entropy of the black hole, respectively, when the backreaction is not considered, and

$$A = 4\pi r_h^2, \qquad v = \frac{Q}{r_h}, \qquad r_h = M_0 + \sqrt{M_0^2 - Q^2}$$

3. DETERMINATION OF THE EQUILIBRIUM TEMPERATURE OF THE SYSTEM

T is the equilibrium temperature of the thermodynamic system we are discussing. What is the relationship between T and T_H?

For the thermodynamic system

$$S = S_0 + S_H + S_r \tag{29}$$

From Gibbons and Hawking (1976)

$$T_{BH} = \left[\left(\frac{\partial S_0}{\partial M_0}\right)_Q\right]^{-1} \tag{30}$$

$$dE_H = T_H\, dS_H + \sigma\, dA \tag{31}$$

Thus

$$dS_H = \frac{dE_H}{T_H} - \frac{\sigma}{T_H}\, dA \tag{32}$$

$$dS_r = \frac{4}{3}\, d(aT_r^3 V) \tag{33}$$

When the system is in thermal equilibrium, we have

$$dE_T = dM_0 + dE_H + d(aT_r^4 V) = 0 \tag{34}$$
$$dV_e = 0$$

Thus

$$dS_T = \left(\frac{1}{T_{BH}} - \frac{1}{T_H}\right) dM_0 + \left(\frac{1}{T_r} - \frac{1}{T_H}\right) d(aT_r^4 V) + \left(\frac{4}{3}\, aT_r^3 - a\,\frac{T_r^4}{T_H} - \frac{\sigma}{T_H}\frac{dA}{dV_H}\right) dV_H \tag{35}$$

where T_{BH}, T_H, and T_R are, respectively, the radiation temperature of the black hole, the two-dimensional thermodynamic membrane, and the radiation field, respectively.

The thermal equilibrum of an isolated system requires that its entropy be maximum, i.e.,

$$dS_T = 0, \qquad dS_T^2 < 0 \tag{36}$$

Thus

$$T_r = T_H = T_{BH} = T \tag{37}$$

$$T = \frac{(M_0^2 - Q^2)^{1/2}}{2\pi r_h^2} \tag{38}$$

where T is the equilibrium temperature of the system.

4. SUMMARY

The existence of the Hawking radiation of a black hole affects space-time, and the black hole can be treated as composed of a naked black hole and a two-dimensional thermodynamic membrane. This black hole and the radiation field compose a thermodynamic system. After considering the back-reaction, we obtain by a thermodynamic approach expressions for the energy and the entropy,

$$M = M_0 + \frac{4}{3}\frac{\lambda a}{(2\pi)^2}\frac{(M_0^2 - Q^2)^{3/2}}{r_h^4} \tag{39}$$

$$S = \pi r_h^2 \left[1 + \frac{\lambda a (M_0^2 - Q^2)}{\pi^2 r_h^4} + a\beta \frac{4}{3}\frac{Q^2}{r_h^4} \right] \tag{40}$$

When $Q = 0$

$$M = M_0 \left[1 + \lambda \frac{4}{3}\frac{a}{(8\pi)^2}\frac{1}{M_0} \right] \tag{41}$$

$$S = 4\pi M_0^2 \left[1 + \frac{4\lambda a}{(8\pi)^2}\frac{1}{M_0^2} \right] \tag{42}$$

When $\lambda = (3/8\pi)C_0$ in (41), our result is in conformity with the energy expresson for a Schwarzschild black hole as obtained by York using a dynamical approach. C_0 is an integration constant.

After considering the backreaction, the energy and entropy of the RNBH are respectively,

$$M = M_0 \left[1 + \frac{C_0 a}{(2\pi)^3}\frac{(M_0^2 - Q^2)^{3/2}}{M_0 r_h^4} \right] \tag{43}$$

$$S = \pi r_h^2 \left[1 + \frac{3C_0 a (M_0^2 - Q^2)}{(2\pi)^3 r_h^4} + a\beta \frac{4}{3}\frac{Q^2}{r_h^4} \right] \tag{44}$$

C_0 is an undefined integration constant in York (1985). The second and third terms in (43) and (44) reflect the energy (mass) and the entropy of the black hole caused by the backreaction of the radiation. Only when M_0 is maximum can the second and the third terms be neglected. In the general case, the backreaction must be considered.

Using a thermodynamic approach, we have obtained expressions for the energy (mass) and the entropy of the RNBH when backreaction is considered, and we have provided an effective method for studying the thermal effects of a black hole and of studying the backreaction problem for more complex black holes.

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